DIFFRACTION OF PLANE SOUND WAVES (LONG) AT A TOROID

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1. We deal with a system of plane sound waves (long waves) moving in the direction x-negative (Fig. 1) and falling on a stationary toroid (of section radius a, the toroid radius being l) whose axis of symmetry coincides with the x-axis of the $Oxyz$-coordinate system.

The velocity potential of sound waves in a simple harmonic system with time multiplier $e^{i\sigma t}$ is defined by the equation

$$\triangle \Phi + k^2\Phi = 0 \qquad \left(k = \frac{\sigma}{c}\right) \tag{1.1}$$

with boundary conditions at the toroid surface Σ in the form

$$\partial\Phi / \partial n = 0 \tag{1.2}$$

where c is the velocity of sound and n is the internal normal to the toroidal surface.

It is required to find a solution to Equation (1.1) in the following form:

$$\Phi = e^{ikx} + \varphi \tag{1.3}$$

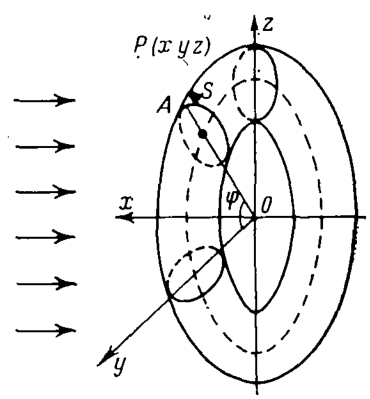

Fig. 1:

where the first term represents the velocity potential of waves approaching along the direction of Ox-negative from infinity, and the second term represents a function of the disturbance caused by waves reflected from the surface of the toroid.

Function ϕ satisfies Equation (1.1).

Let us establish the boundary conditions for function ϕ. To do this we have to introduce a new coordinate system r', θ, ψ as shown in Fig. 2 and which is related to the rectangular xyz-system by the following expressions:

$$x = r' \sin\theta, \qquad y = (l + r' \cos\theta) \cos\psi, \qquad z = (l + r'\cos\theta) \sin\psi \tag{1.4}$$

The equations of the toroid surface in this system are

$$x = a \sin\theta, \qquad y = (l + a\cos\theta)\cos\psi, \qquad z = (l + a\cos\theta)\sin\psi \tag{1.5}$$

As the wavelength λ is large compared with the toroid section radius a (i.e. k is small) it is possible to expand e^{ikx} close to the toroid surface in a Taylor series. We then have

$$e^{ikx} = 1 + kir' \sin\theta - \frac{k^2 r'^2 \sin^2\theta}{2} + \cdots \tag{1.6}$$

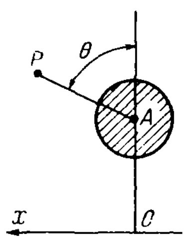

Fig. 2.

On the toroid surface (Σ) we have

$$\partial\varphi/\partial n = -\,\partial\varphi/\partial r' \quad \text{for } r' = a \tag{1.7}$$

On a basis of (1.1), (1.2), (1.6) and (1.7) function ϕ satisfies the boundary condition

$$\frac{\partial\varphi}{\partial n} = -\frac{\partial}{\partial r'}\left(-e^{ikx}\right) = -[-ki\sin\theta - k^2 r' \sin^2\theta + \ldots]_{r'=a}$$
$$= +ki\sin\theta - k^2 a \sin^2\theta + \ldots \tag{1.8}$$

If we take account of boundary condition (1.8), the solution of (1.4) can be expressed as a surface integral, i.e. the velocity potential of the reflected waves at point P can be determined by the formula [1]

$$\varphi(P) = \frac{1}{4\pi}\iint_{\Sigma}\left[\varphi\frac{\partial}{\partial n}\frac{e^{-ik\rho}}{\rho} - \frac{e^{-ik\rho}}{\rho}\frac{\partial\varphi}{\partial n}\right]d\sigma \tag{1.9}$$

In this expression ρ is the distance between the fixed point P lying outside Σ and any arbitrary point on the toroid surface Σ.

As the function ϕ on the toroid surface is unknown, it is not yet possible to determine $\phi(P)$ from Expression (1.9). In the following two sections we will therefore discuss the determination of function ϕ on the toroid surface Σ.

2. Assume that on the toroid surface and in its immediate vicinity the function can be expanded in the series

$$\varphi = \varphi_0 + k\varphi_1 + k^2\varphi_2 + \ldots, \tag{2.1}$$

where ϕ_0, ϕ_1, ϕ_2 are as yet unknown. Substitute this expansion in (1.4) (it can be done on the basis of (1.8)) and, on equating coefficients of terms in similar powers on both sides of the identities, we arrive at the differential equations

$$\Delta\varphi_0 = 0, \qquad \Delta\varphi_1 = 0, \qquad \Delta\varphi_2 + \varphi_0 = 0 \quad \text{etc.} \tag{2.2}$$

It follows from (1.8) and (2.1) that on the toroid surface the functions ϕ_0, ϕ_1, ϕ_2, ... will satisfy, respectively, the following boundary conditions;

$$\frac{\partial\varphi_0}{\partial n} = 0, \qquad \frac{\partial\varphi_1}{\partial n} = i\sin\theta, \qquad \frac{\partial\varphi_2}{\partial n} = -a\sin^2\theta \tag{2.3}$$

In order to determine the solution to (1.1) for long sound waves to an accuracy of $O(k^3)$ it is necessary, in order to determine ϕ on the toroid surface, to solve the three first differential equations of system (2.2) with the corresponding boundary conditions at Σ. Then function ϕ will take the form

$$\varphi = \varphi_0 + k\varphi_1 + k^2\varphi_2 + O(k^3) \tag{2.4}$$

3. It is well known that in the curvilinear orthogonal coordinate system x_1, x_2, x_3 the operator $\Delta\phi$ is determined by the formula

$$\Delta\varphi = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial x_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial\varphi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial\varphi}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1 h}{h_3}\frac{\partial\varphi}{\partial x_3}\right)\right\} \tag{3.1}$$

Here h_1, h_2, h_3 are metric or Lamé coefficients. The orthogonal coefficients $x_1 = r'$, $x_2 = \theta$, $x_3 = \psi$ are connected with the rectangular coordinates of Equations (1.4).

The metric coefficients are, respectively

$$h_1 = 1, \qquad h_2 = r', \qquad h_3 = l + r'\cos\theta$$

Thus Formula (3.1) takes the following form:

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial r'^2} + \frac{1}{r'^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{(l + r'\cos\theta)^2}\frac{\partial^2\varphi}{\partial\psi^2} + \frac{l + 2r'\cos\theta}{r'(l + r'\cos\theta)}\frac{\partial\varphi}{\partial r'} - \frac{\sin\theta}{r'(l + r'\cos\theta)}\frac{\partial\varphi}{\partial\theta} \tag{3.2}$$

Owing to the symmetry of the toroid about the Ox-axis, the functions ϕ_0 and ϕ_1 will be independent of ψ. Thus the first of the two Laplace equations from (2.2) becomes

$$\frac{\partial^2\varphi_0}{\partial r'^2} + \frac{1}{r'^2}\frac{\partial^2\varphi_0}{\partial\theta^2} + \frac{l + 2r'\cos\theta}{r'(l + r'\cos\theta)}\frac{\partial\varphi_0}{\partial r'} - \frac{\sin\theta}{r'(l + r'\cos\theta)}\frac{\partial\varphi_0}{\partial\theta} = 0 \tag{3.3}$$

$$\frac{\partial^2\varphi_1}{\partial r'^2} + \frac{1}{r'^2}\frac{\partial^2\varphi_1}{\partial\theta^2} + \frac{l + 2r'\cos\theta}{r'(l + r'\cos\theta)}\frac{\partial\varphi_1}{\partial r'} - \frac{\sin\theta}{r'(l + r'\cos\theta)}\frac{\partial\varphi_1}{\partial\theta} = 0 \tag{3.4}$$

Equation (3.3) with boundary condition $\partial\phi_0/\partial n = 0$ has the unique solution

$$\varphi_0 = \text{const} = c \tag{3.5}$$

We will look for a solution to Equation (3.4) with boundary condition $\partial\phi_1/\partial n = i \sin\theta$ in the form $\phi_1 = - ir' \sin\theta$.

It can be found by direct verification that function ϕ_1 satisfies Equation (3.4) and the condition $\partial\phi_1/\partial n = i \sin\theta$ on the toroid surface Σ. As $r = a$ on the toroid surface we have

$$\varphi_1 = - ia \sin\theta \tag{3.6}$$

Because of (3.2) and (3.5) and the symmetry of the toroid, the equation for ϕ_2 from (2.2) takes on the following form:

$$\frac{\partial^2\varphi_2}{\partial r'^2} + \frac{1}{r'^2}\frac{\partial^2\varphi_2}{\partial\theta^2} + \frac{l + 2r'\cos\theta}{r'(l + r'\cos\theta)}\frac{\partial\varphi_2}{\partial r'} - \frac{\sin\theta}{r'(l + r'\cos\theta)}\frac{\partial\varphi_2}{\partial\theta} + c = 0 \tag{3.7}$$

with boundary condition

$$\partial\varphi_2/\partial n = -a\sin^2\theta \tag{3.8}$$

We look for a solution to Equation (3.7) for the case when $c = - 1$. It is possible to show by direct verification that the function $\phi_2 = + 1/2\ r'^2 \sin^2\theta$ satisfies Equation (3.7) and the condition (3.8). We have on the surface of the toroid

$$\varphi_2 = + \frac{1}{2}a^2\sin^2\theta \tag{3.9}$$

Because of (2.4), (3.5), (3.6) and (3.9) function ϕ on Σ has the following value:

$$\varphi = -\Big(1 + kia\sin\theta - k^2\frac{a^2\sin^2\theta}{2}\Big) + O(k^3) \tag{3.10}$$

4. Assuming that ϕ and $\partial\phi/\partial n$ on Σ are in accordance with (3.10) and (1.8), we obtain from (1.9)

$$\varphi(P) = -\frac{1}{4\pi}\iint_{\Sigma}\Big[\Big(1 + kia\sin\theta - \frac{k^2a^2\sin^2\theta}{2}\Big)\frac{\partial}{\partial n}\frac{e^{-ik\rho}}{\rho} + (ki\sin\theta - k^2 a\sin^2\theta)\frac{e^{-ik\varphi}}{\rho}\Big]d\sigma + O(k^3) \tag{4.1}$$

From Formula (4.1) it is possible to determine the velocity potential of the reflected waves for any point $P(x, y, z)$, lying outside the toroid.

We will now find an approximate value of integral (4.1) for the case where distances ρ are large as compared with the dimensions of the toroid.

Let $OM = R$, $MR = \rho$, $OP = R_0$, $\angle YOC = \psi$ and $DCM = \theta$, where M is an arbitrary variable point lying on the toroid surface with coordinates (1.5) (Fig. 3).

Let us write down the following approximation

$$\frac{e^{-ik\rho}}{\rho} = \left[1 - x_1\frac{\partial}{\partial x} - y_1\frac{\partial}{\partial y} - z_1\frac{\partial}{\partial z}\right]\frac{e^{-ikR_0}}{R_0} + \ldots \quad (4.2)$$

where $R = OP = \sqrt{(x^2 + y^2 + z^2)}$, the distance of point P from the origin. From (4.2) we have

$$\frac{\partial}{\partial n}\frac{e^{-ik\rho}}{\rho} = -\left(l_1\frac{\partial}{\partial x} + m_1\frac{\partial}{\partial y} + n_1\frac{\partial}{\partial r}\right)\frac{e^{-ikR_0}}{R_0} + \ldots \quad (4.3)$$

Here l_1, m_1, n_1 are the direction cosines of the external normal. We have

$$l_1 = -\sin\theta,\ m_1 = -\cos\theta\cos\psi,\ n_1 = -\cos\theta\sin\psi \quad (4.4)$$

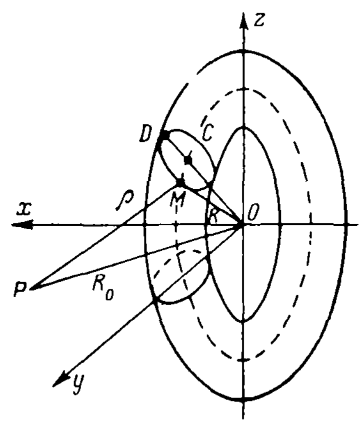

Fig. 3.

The elementary area $d\sigma$ on the toroid surface is equal to

$$d\sigma = a\,(l + a\cos\theta)\,d\theta d\psi \quad (4.5)$$

On a basis of (4.2) to (4.5) therefore, Formula (4.1) takes on the form

$$\varphi(P) = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\Big[\Big(1 + kia\sin\theta - \frac{k^2 a\sin^2\theta}{2}\Big)\Big(\sin\theta\frac{\partial}{\partial x} + \cos\psi\frac{\partial}{\partial y} + \cos\theta\sin\psi\frac{\partial}{\partial z}\Big)\frac{e^{ikR_0}}{R_0} - (ki\sin\theta - k^2 a\sin^2\theta)\Big(1 - \sin\theta\frac{\partial}{\partial x} - \cos\theta\cos\psi\frac{\partial}{\partial y} - \cos\theta\sin\psi\frac{\partial}{\partial z}\Big)\frac{e^{-ikR_0}}{R_0}\Big]a\,(l + a\cos\theta)\,d\theta d\psi + O(k^3) \quad (4.6)$$

or

$$\varphi(P) = -\frac{kQi}{2\pi}\frac{\partial}{\partial x}\frac{e^{-ikR_0}}{R_0} - \frac{k^2 Q}{4\pi}\frac{e^{-ikR_0}}{R_0} + O(k^3) \quad (4.7)$$

where $Q = 2\pi^2 a^2 l$ is the toroid volume.

It follows from Formula (4.7) that the reflected waves can be regarded as arising from the combined action of a simple source and a dipole situated at the origin of the *Oxyz*-coordinate system.

Having now found ϕ for any given point P lying some distance from the toroid, it is possible to write down a full expression for the velocity potential Φ taking into account the time multiplier and sound intensity J_0 in the form

$$\Phi = J_0 e^{i(kx+\sigma t)} + J_0 \frac{kQ}{2\pi} \frac{\partial}{\partial x} \frac{e^{i(\sigma t - kR_0 - \pi)}}{R_0} + J_0 \frac{k^2 Q}{4\pi} \frac{e^{i(\sigma t - kR_0 - \pi)}}{R_0} + Q(k^3) \tag{4.8}$$

Evaluating the real part, we have

$$\Phi = J_0 \cos(kx + \sigma t) + J_0 \frac{kQ}{4} \frac{\partial}{\partial x} \frac{\cos(\sigma t - kR_0 - \pi)}{R_0} + \\ + J_0 \frac{k^2 Q}{4} \frac{\cos(\sigma t - kR_0 - \pi)}{R_0} + O(k^3) \tag{4.9}$$

$$(R_0 = \sqrt{x^2 + y^2 + z^2})$$

It is evident from Formula (4.9) that the velocity potential of the sound wave at any point located at a large distance from the toroid equals the sum of three velocity potentials. The first of these is the velocity potential of a plane sound wave originating at infinity and travelling at constant velocity σ/k in the negative Ox-direction. The second one is from the dipole at the origin and the third is from a point source also situated at the origin.

Thus the problem posed in Section 1 is solved by Formula (4.9).

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